

Find the curvature of $\vec{r}(t) = te^t \vec{i} - 2\cos t \vec{j} + te^{-t} \vec{k}$ at the point $(0, -2, 0)$. $te^t = 0 \rightarrow t=0$

SCORE: ____ / 20 PTS

$$\vec{r}'(t) = \langle e^t + te^t, 2\sin t, e^{-t} - te^{-t} \rangle \quad \vec{r}'(0) = \langle 1, 0, 1 \rangle$$

$$\vec{r}''(t) = \langle 2e^t + te^t, 2\cos t, -e^{-t} + te^{-t} \rangle \quad \vec{r}''(0) = \langle 2, 2, -2 \rangle$$

$$K(0) = \frac{\|\langle 1, 0, 1 \rangle \times \langle 2, 2, -2 \rangle\|}{\|\langle 1, 0, 1 \rangle\|^3} = \frac{\|\langle -2, 4, 2 \rangle\|}{\sqrt{2}^3}$$

$$= \frac{2\|\langle -1, 2, 1 \rangle\|}{2\sqrt{2}}$$

$$= \frac{\sqrt{6}}{\sqrt{2}}$$

$$= \sqrt{3}$$

Find a vector function for the curve of intersection of the surfaces $x^2 + y^2 + z^2 = 34$ and $x - z = 6$.

SCORE: ____ / 25 PTS

$$x = z + 6$$

$$(z+6)^2 + y^2 + z^2 = 34 \quad (5)$$

$$2z^2 + 12z + y^2 = -2$$

$$2(z^2 + 6z + 9) + y^2 = -2 + 18 = 16$$

$$2(z+3)^2 + y^2 = 16$$

$$\frac{(z+3)^2}{8} + \frac{y^2}{16} = 1 \quad (8)$$

$$\frac{(z+3)^2}{8} = \cos^2 t \quad \frac{y^2}{16} = \sin^2 t$$

$$(3) \quad z = -3 + 2\sqrt{2} \cos t, \quad y = 4 \sin t \quad (2)$$

$$(3) \quad x = 3 + 2\sqrt{2} \cos t$$

$$\vec{r}(t) = \langle 3 + 2\sqrt{2} \cos t, 4 \sin t, -3 + 2\sqrt{2} \cos t \rangle \quad (3)$$

Consider the graph of $y = 3 \ln x$.

SCORE: ____ / 20 PTS

- [a] Find the curvature at the point $(4, 3 \ln 4)$.

$$f'(x) = 3x^{-1} \quad f'(4) = \frac{3}{4}$$

$$f''(x) = -3x^{-2} \quad f''(4) = -\frac{3}{16}$$

$$K(4) = \frac{\left|1 - \frac{3}{16}\right|}{\sqrt{1 + \left(\frac{3}{4}\right)^2}}^3 = \frac{\frac{3}{16}}{\left(\frac{5}{4}\right)^3} = \frac{3}{16} \cdot \frac{4^3}{5^3} = \frac{12}{125}$$

- [b] **BONUS QUESTION (5 POINTS):**

Suppose $y = f(x)$ is a function which is infinitely differentiable (ie. all derivatives exist at all points in the domain).

Explain why the inflection points of f are points of minimum curvature. Justify your answer algebraically.

$K \geq 0$ f'' @ INFLECTION POINT = 0 SINCE f'' EXISTS EVERYWHERE
SO $K = 0$ @ INFLECTION POINT, IE. MINIMUM K

Let $\vec{u}(t) = \vec{r}(t) \cdot [\vec{r}'(t) \times \vec{w}(t)]$.

SCORE: ____ / 25 PTS

- [a] Find an expression for $\vec{u}'(t)$.

NOTE: Your final answer must be completely simplified.

It may use the derivatives of vector functions, but it must NOT use the derivatives of dot and cross products.

$$\vec{u}'(t) = \underbrace{\vec{r}'(t) \cdot [\vec{r}'(t) \times \vec{w}(t)]}_{(5)} + \vec{r}(t) \cdot \underbrace{[\vec{r}''(t) \times \vec{w}(t) + \vec{r}'(t) \times \vec{w}'(t)]}_{(5)} \quad (5)$$

BUT $\vec{r}'(t) \cdot [\vec{r}'(t) \times \vec{w}(t)] = 0$ SINCE CROSS PRODUCT IS ORTHOGONAL TO ORIGINAL VECTORS
 (5)

$$\text{SO } \vec{u}'(t) = \vec{r}(t) \cdot [\vec{r}''(t) \times \vec{w}(t) + \vec{r}'(t) \times \vec{w}'(t)]$$

(3)

- [b] In addition, suppose $\vec{w}(t)$ is a function such that $\vec{w}(t)$ and $\vec{r}''(t)$ always point in exact opposite directions.
 Find an expression for $\vec{u}'(t)$.

NOTE: Your final answer must be completely simplified.

$$\|\vec{r}''(t) \times \vec{w}(t)\| = \|\vec{r}''(t)\| \|\vec{w}(t)\| \sin 180^\circ = 0$$

$$\text{SO } \vec{r}''(t) \times \vec{w}(t) = \vec{0} \quad (5)$$

$$\text{SO } \vec{u}'(t) = \vec{r}(t) \cdot [\vec{r}'(t) \times \vec{w}'(t)] \quad (2)$$

Consider the curve $\vec{r}(t) = \langle \sin t - t \cos t, 1 - t^2, \cos t + t \sin t \rangle$.

SCORE: ____ / 60 PTS

[a] Find the unit tangent vector $\vec{T}(t)$.

$$\vec{r}'(t) = \langle t \sin t, -2t, t \cos t \rangle = t \langle \sin t, -2, \cos t \rangle$$

$$\|\vec{r}'(t)\| = |t| \sqrt{\sin^2 t + 4 + \cos^2 t} = \sqrt{5} |t|$$

$$\vec{T}(t) = \frac{1}{\sqrt{5} |t|} t \langle \sin t, -2, \cos t \rangle$$

$$= \begin{cases} -\frac{1}{\sqrt{5}} \langle \sin t, -2, \cos t \rangle, & t < 0 \\ \frac{1}{\sqrt{5}} \langle \sin t, -2, \cos t \rangle, & t > 0 \end{cases}$$

IF YOU ASSUMED THAT $t > 0$
AND ONLY GOT $\frac{1}{\sqrt{5}} \langle \sin t, -2, \cos t \rangle$,
YOU WILL GET FULL CREDIT

[b] Find parametric equations for the tangent line to the curve at the point $(\pi, 1 - \pi^2, -1)$.

$$\vec{r}'(\pi) = \pi \langle 0, -2, -1 \rangle$$

$$1 - t^2 = 1 - \pi^2 \rightarrow t = \pi, -\pi$$

$$\text{USE } \langle 0, 2, 1 \rangle$$

$$\begin{matrix} \downarrow & \downarrow \\ x = \pi & x = -\pi \end{matrix}$$

$$\begin{cases} x = \pi \\ y = 1 - \pi^2 + 2t \\ z = -1 + t \end{cases}$$

[c] Find the curvature function $\kappa(t)$.

$$\vec{T}'(t) = \begin{cases} -\frac{1}{\sqrt{5}} \langle \cos t, 0, -\sin t \rangle, & t < 0 \\ \frac{1}{\sqrt{5}} \langle \cos t, 0, -\sin t \rangle, & t > 0 \end{cases}$$

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{1}{\sqrt{5}} \sqrt{\cos^2 t + \sin^2 t}}{\sqrt{5} |t|} = \frac{1}{5 |t|}$$

[d] Find the length of the curve from the point $(\pi, 1 - \pi^2, -1)$ to the point $(-2\pi, 1 - 4\pi^2, 1)$.

$$\int_{\pi}^{-2\pi} \|\vec{r}'(t)\| dt = \int_{\pi}^{-2\pi} \sqrt{5} |t| dt = \frac{\sqrt{5}}{2} t^2 \Big|_{\pi}^{-2\pi} = \frac{\sqrt{5}}{2} (4\pi^2 - \pi^2) = \frac{3\sqrt{5}}{2} \pi^2$$